TOPOLOGY PROCEEDINGS

Volume 7, 1982

Pages 109–118

http://topology.auburn.edu/tp/

PRIMALITY OF CERTAIN KNOTS

by

Kenneth A. Perko, Jr.

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

PRIMALITY OF CERTAIN KNOTS

Kenneth A. Perko, Jr.

This paper proves that the fifteen 4-bridged examples in J. H. Conway's table of 11-crossing knots [2] are each actually prime. Note that we avoid reliance upon assumptions that the prime knot tables are complete or that the minimal crossing number is additive. Cf. [8] and [3].

Reproduced herewith are diagrams of the 552 known 11-crossing primes, of which we here consider knots 12, 84, 220, 225, 240, 357, and 426 through 434. Proof of their primality by another method appears in [4].

Proposition. A knot is prime if (1) *its bridge number* $b(k) \leq 4$ *and* (2) *with respect to the homology of its* 3*-fold dihedral covering spaces* (a) *no* $H_1M_3(k)$ *has odd order and* (b) *the orders of the* $H_1M_3(k)$*'s have no common factor* >3.

Remarks. The first condition may be verified by finding four generators of an entire knot diagram. Compare [1] and [6, p. 606] but beware the concealed conjecture in the latter that bridge number *equals* the minimal number of Wirtinger generators. The second condition must be verified by calculation of $H_1M_3(k)$ for all possible representations of the knot group on S_3 [5]. In the case of each of our 15 examples we get two homology groups, Z_6 and $Z+Z_2$. Note that the existence of a single noncyclic $H_1M_3(k)$ implies that $b(k) > 3$ [1].

Proof. It follows from condition (1) and Schubert's Satz 7 [7]--$b(k_1 \# k_2) = b(k_1)+b(k_2)-1$--that k is prime unless it has a 2-bridged factor. Assume, arguendo, that $k = (\alpha,\beta) \# k'$, where (α,β) is Schubert's normal form notation for 2-bridged knots. Recall that α is odd and >1. Either 3 divides α or 3 does not divide α. We shall derive a contradiction from each of these two possibilities.

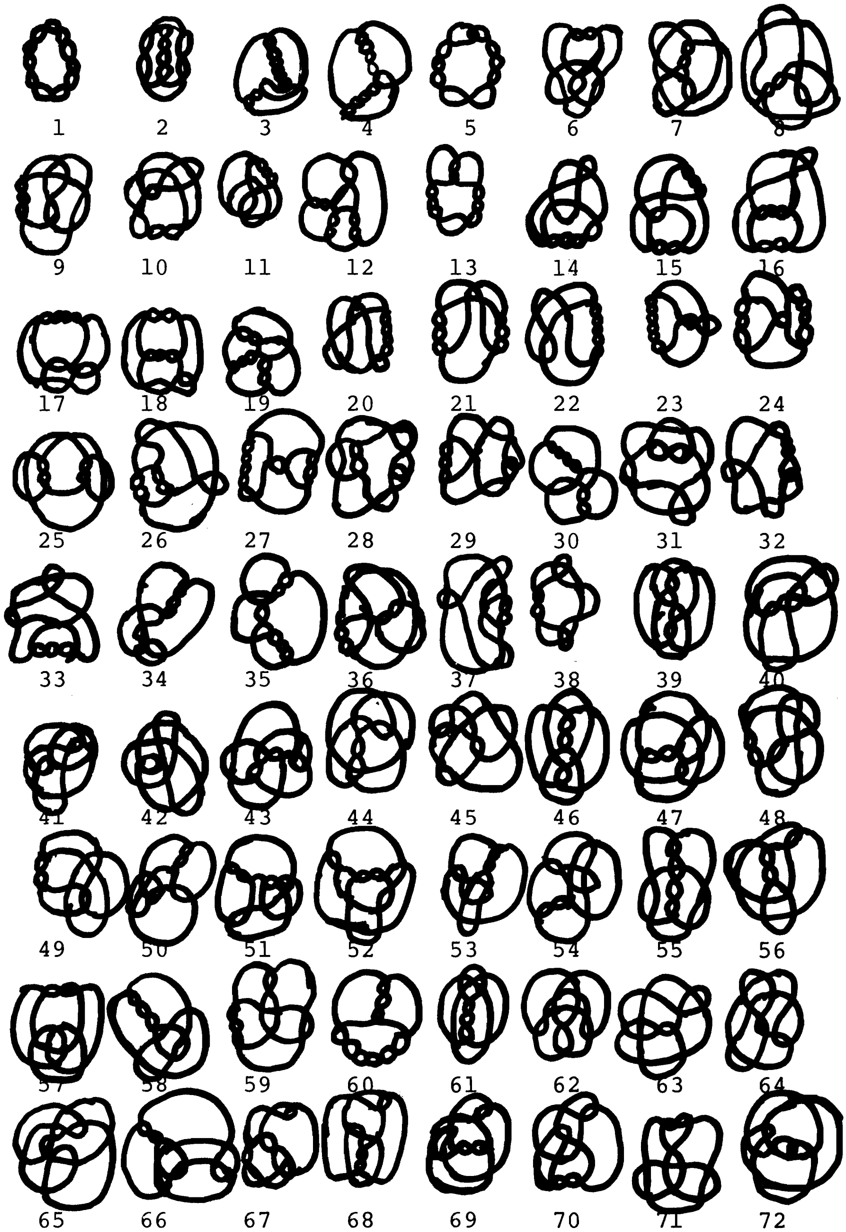
Case A. $3|\alpha$. Then k has a 3-fold irregular covering obtained from the S_3 cover of (α,β) and the constant map on the k' factor--i.e., that which sends all meridians to a single transposition. The homology of such a cover is the same as that of the double branched cover of the knot k'. But the order of the latter, $|H_1M_2(k')| = \Delta(-1)(k')$, is well known to be odd, which violates condition (2)(a).

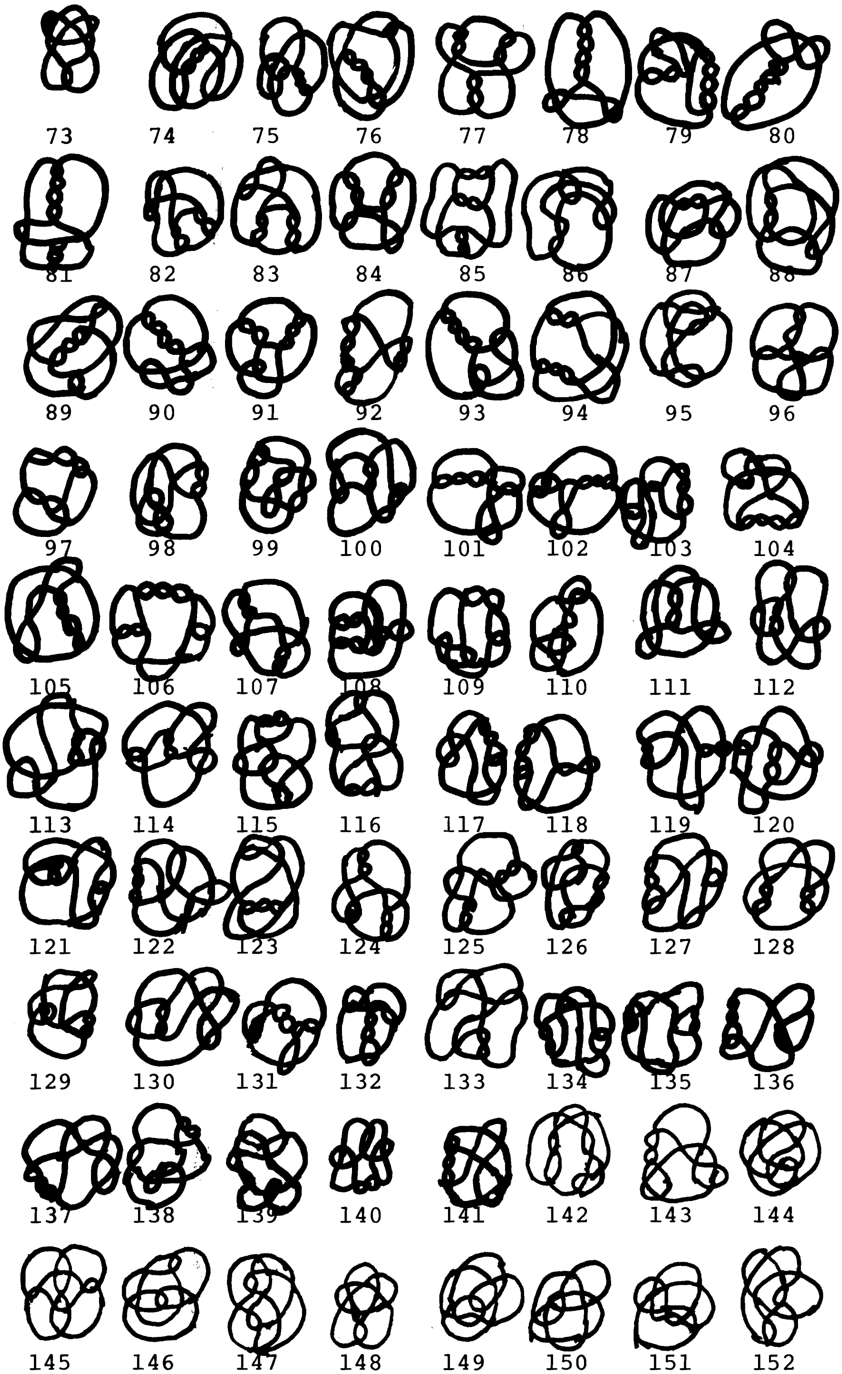
Case B. $3\nmid\alpha$. Here every 3-fold irregular cover of k must be the constant map on the (α,β) factor. By a suitable adjoining of relations derived from the k' factor each such $H_1M_3(k)$ can be shown to admit a surjection on the homology of the 2-fold cyclic cover of (α,β), $H_1M_2(\alpha,\beta) = Z_\alpha$. But this contradicts condition (2)(b). (Indeed, in the case of our 15 examples we get $Z_6 \twoheadrightarrow Z_\alpha$ and $Z+Z_2 \twoheadrightarrow Z_\alpha$ which together imply that $\alpha = 3$.)

Thus k can have no 2-bridged factor and is therefore prime.

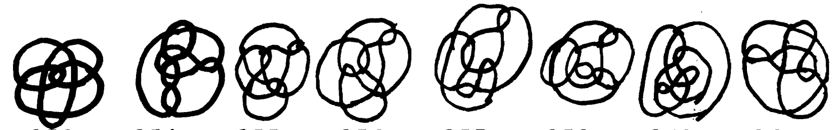
PRIME KNOTS WITH ELEVEN CROSSINGS

Drawn by Kathryn Perko

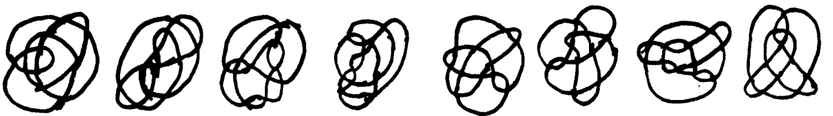




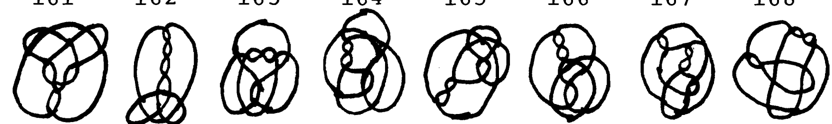
73 74 75 76 77 78 79 80
81 82 83 84 85 86 87 88
89 90 91 92 93 94 95 96
97 98 99 100 101 102 103 104
105 106 107 108 109 110 111 112
113 114 115 116 117 118 119 120
121 122 123 124 125 126 127 128
129 130 131 132 133 134 135 136
137 138 139 140 141 142 143 144
145 146 147 148 149 150 151 152



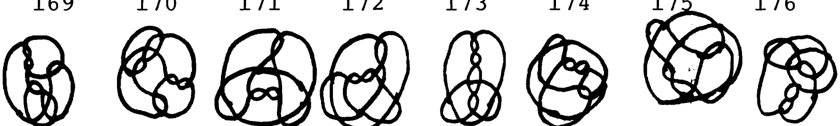
153 154 155 156 157 158 159 160



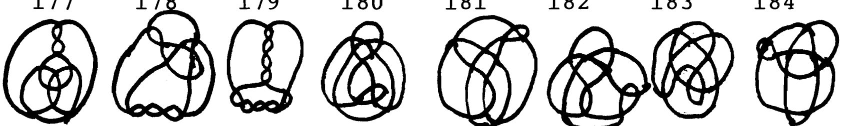
161 162 163 164 165 166 167 168



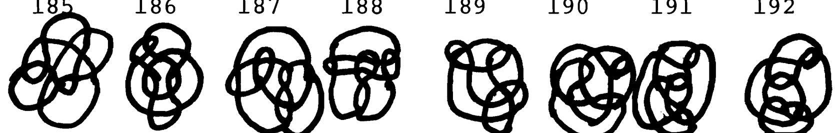
169 170 171 172 173 174 175 176



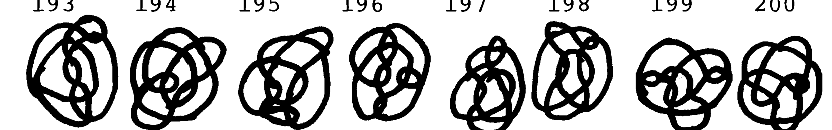
177 178 179 180 181 182 183 184



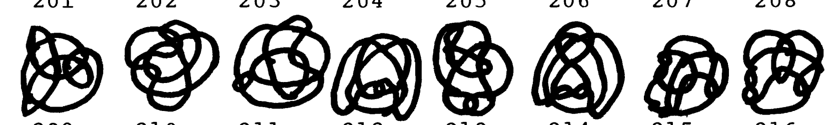
185 186 187 188 189 190 191 192



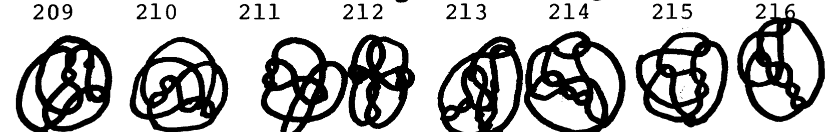
193 194 195 196 197 198 199 200



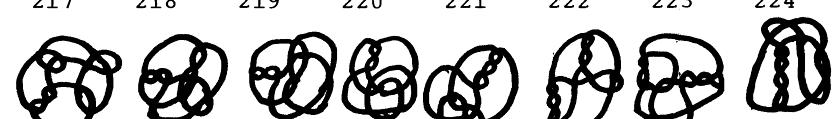
201 202 203 204 205 206 207 208



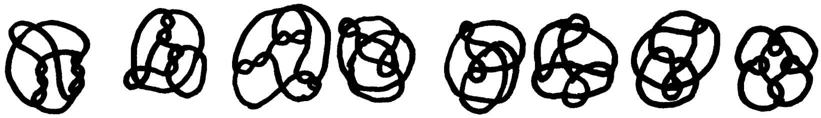
209 210 211 212 213 214 215 216



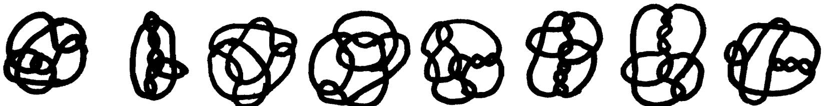
217 218 219 220 221 222 223 224



225 226 227 228 229 230 231 232

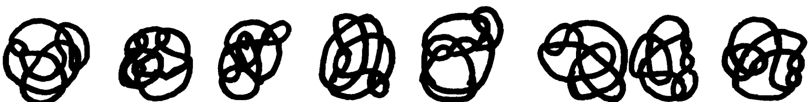


233 234 235 236 237 238 239 240

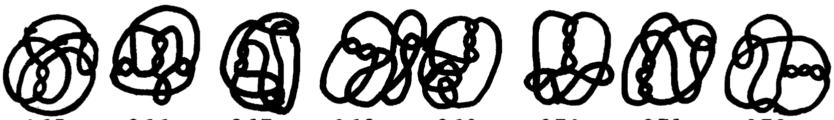


241 242 243 244 245 246 247 248

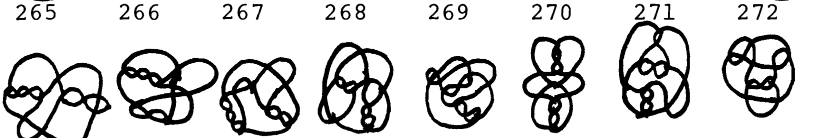
249 250 251 252 253 254 255 256



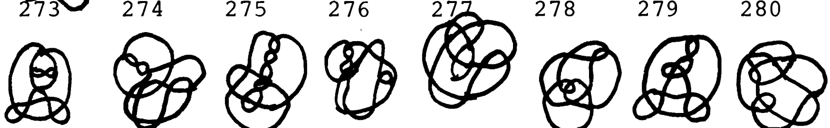
257 258 259 260 261 262 263 264



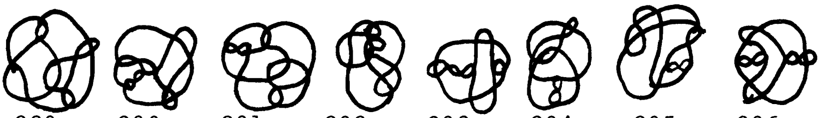
265 266 267 268 269 270 271 272



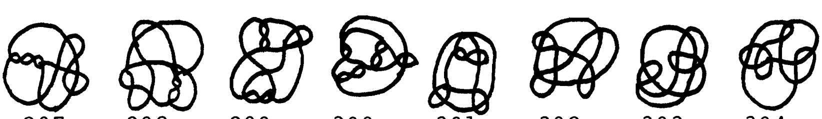
273 274 275 276 277 278 279 280



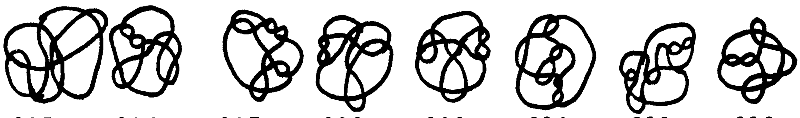
281 282 283 284 285 286 287 288



289 290 291 292 293 294 295 296



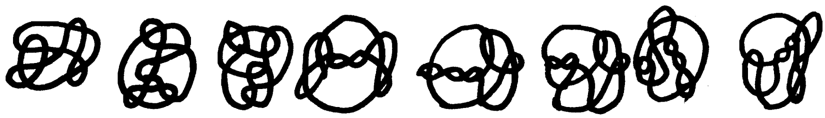
297 298 299 300 301 302 303 304



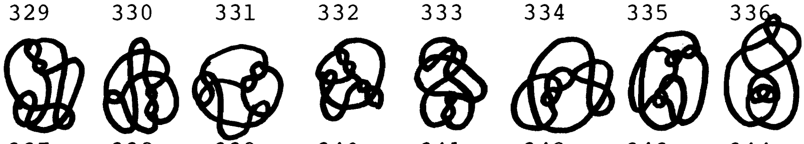
305 306 307 308 309 310 311 312

313 314 315 316 317 318 319 320

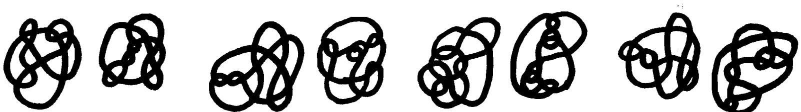
321 322 323 324 325 326 327 328



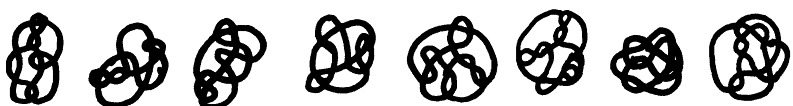
329 330 331 332 333 334 335 336



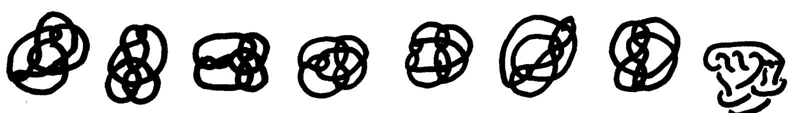
337 338 339 340 341 342 343 344



345 346 347 348 349 350 351 352



353 354 355 356 357 358 359 360

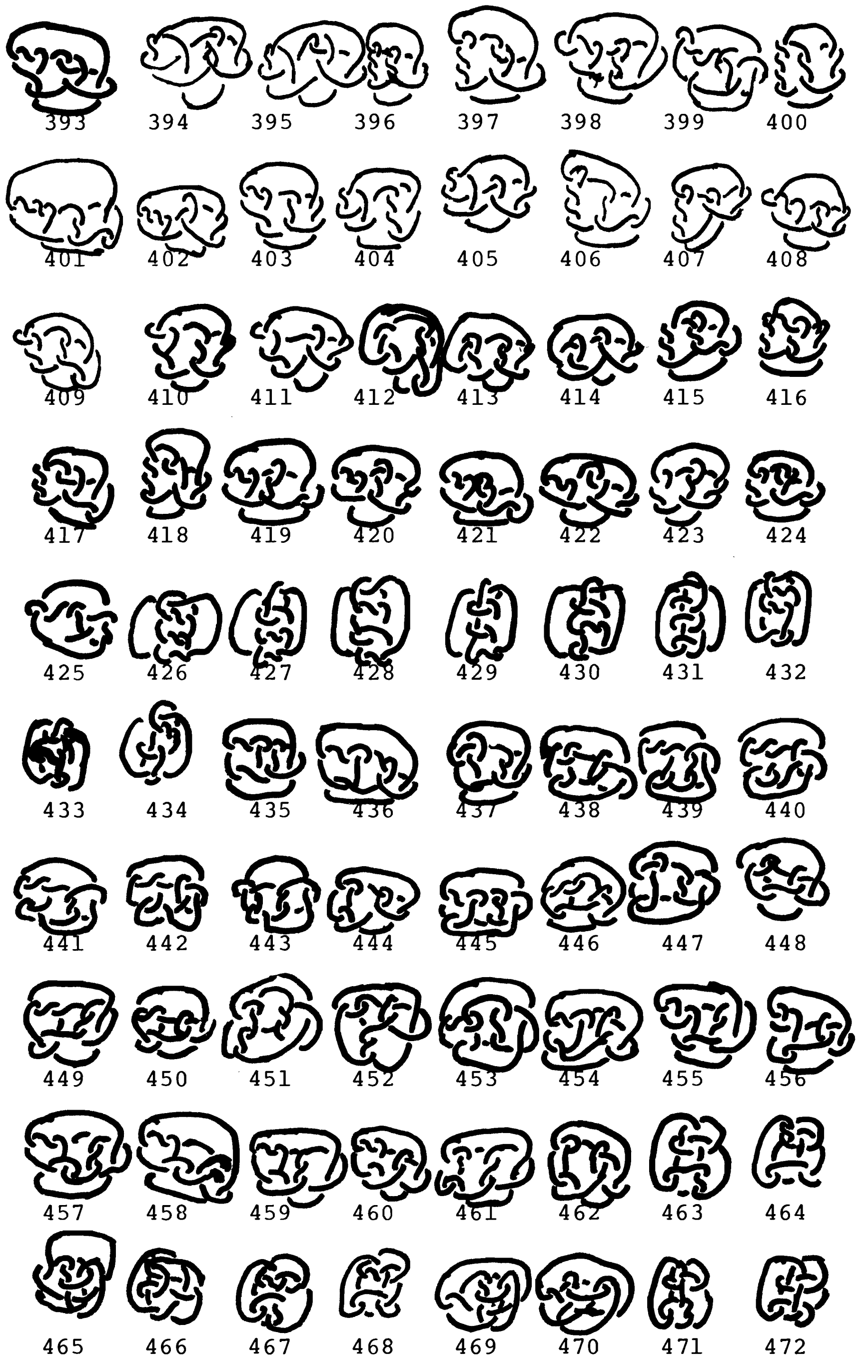


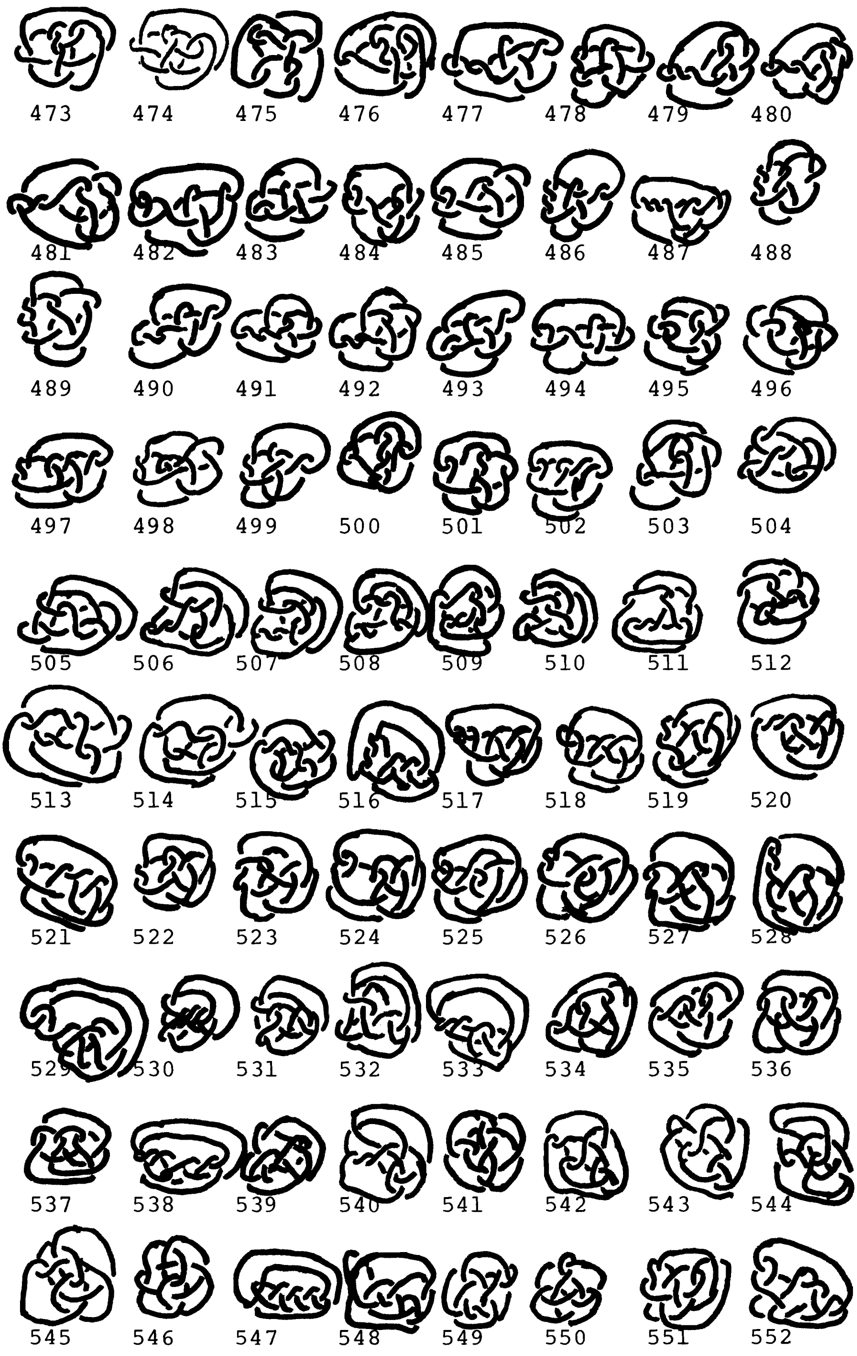
361 362 363 364 365 366 367 368

369 370 371 372 373 374 375 376

377 378 379 380 381 382 383 384

385 386 387 388 389 390 391 392


393 394 395 396 397 398 399 400
401 402 403 404 405 406 407 408
409 410 411 412 413 414 415 416
417 418 419 420 421 422 423 424
425 426 427 428 429 430 431 432
433 434 435 436 437 438 439 440
441 442 443 444 445 446 447 448
449 450 451 452 453 454 455 456
457 458 459 460 461 462 463 464
465 466 467 468 469 470 471 472


473 474 475 476 477 478 479 480
481 482 483 484 485 486 487 488
489 490 491 492 493 494 495 496
497 498 499 500 501 502 503 504
505 506 507 508 509 510 511 512
513 514 515 516 517 518 519 520
521 522 523 524 525 526 527 528
529 530 531 532 533 534 535 536
537 538 539 540 541 542 543 544
545 546 547 548 549 550 551 552

References

[1] G. Burde, *On branched coverings of* S^3, Canad. J. Math. 23 (1971), 84-89.

[2] J. H. Conway, *An enumeration of knots and links, and some of their algebraic properties*, Computational problems in abstract algebra, Prof. Conf. Oxford, 1967, 329-358 (Pergamon Press, Oxford, 1970).[1]

[3] C. McA. Gordon. *Problem* 1, *Knot theory*, Lecture notes in mathematics 685, Springer-Verlag, 1978, 309.

[4] W. B. Raymond Lickorish, *Prime knots and tangles*, Trans. Amer. Math. Soc. 267 (1981), 321-332.

[5] K. A. Perko, Jr., *On dihedral covering spaces of knots*, Inventiones Math. 34 (1976), 77-82.

[6] R. Riley, *Homomorphisms of knot groups on finite groups*, Math. Comp. 25 (1971), 603-619.

[7] H. Schubert, *Über eine numerische Knoteninvariante*, Math. Zeit. 61 (1954), 245-288.

[8] J. M. Van Buskirk, *A class of -amphicheiral knots and their Alexander polynomials*, Preprint Series 1977/78, No. 18, Matematisk Institut, Aarhus Universitet, 1978.

---

[1]Note that Conway's 11-crossing table omits knot types 549 through 552. Several thousand 12-crossing knots have recently been classified by Morwen B. Thistlethwaite of the Polytechnic of the South Bank.

Rockefeller Center, Inc.

1230 Avenue of the Americas

New York, New York 10020-1579